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A METHOD OF CALCULATING BENDING STRESSES DUE TO TORSION

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ADVANCE-RESTRICTED REPORT

A METHOD OF CALCULATING BENDING STRESSES DUE TO TORSION

By Paul Kuhn

SUMMARY

A method is described for analyzing bending stresses due to torsion in a box with variable cross section and loading by means of a recurrence formula leading to a set of equations identical in form with the well-known three-moment equations. Formulas are given to deal with special features such as full-width cut-outs and carry-through bays. In conclusion, an approximate form of the general method of analysis is presented that eliminates the need of solving a system of equations. The simplified method is sufficiently accurate for most requirements of practical stress analysis and may be applied without knowledge of the general method. Numerical examples illustrate all procedures.

INTRODUCTION

The basic strength element of many wings is a box of approximately rectangular cross section. When such a box is loaded by torques, the walls are subjected to shear stresses that can be calculated by the well-known formula for shells in torsion. In addition to the primary system of shear stresses, localized systems of secondary stresses are set up in the vicinity of concentrated torques and of discontinuities of the cross-sectional dimensions. These secondary stress systems are frequently referred to as bending stresses due to torsion, because their resultants are bending moments in the planes of the walls, accompanied by the shear forces necessary to cause spanwise variations of the bending moments.

In actual wing structures, the spanwise variations of loading and of cross-sectional dimensions cannot be represented very well by simple mathematical expressions. General methods of calculating bending stresses due to torsion therefore use the familiar procedure of dividing the box into bays such that the cross-sectional dimensions

and the torques may be assumed constant within each bay. The foundation for such general methods was laid by Ebner in a comprehensive paper (reference 1); subsequent authors have followed Ebner's lead more or less closely.

Numerical calculations made by Ebner and others lead to the conclusion that bending stresses due to torsion are of practical importance only when the discontinuities of the loading or of the dimensions are very marked. The distribution of the loading and the dimensions of the cross sections at a distance from the discontinuity have only a negligible influence on the stresses at the discontinuity. Advantage could be taken of these facts to reduce the numerical work required for estimating the maximum stresses in the vicinity of major discontinuities if formulas were available free of Ebner's assumption that no ribs exist within a bay. Such formulas are developed in this paper by a method that combines parts of the methods of Ebner (reference 1), Reissner (reference 2), and Orszedziolski (reference 3).

For the final stress analysis it may be desirable to divide the beam into short bays; numerical difficulties may then be encountered in the application of the formulas. Parallel formulas are derived, therefore, based on the assumption that the bays are very short. This assumption leads to the same results as the assumption that no ribs exist within the bays, which is Ebner's basic assumption, and the formulas represent a special case of Ebner's theory.

ANALYSIS OF A BOX WITHOUT CUT-OUTS

General Considerations

Synopsis of problem and procedure.— The problem to be investigated may be stated as follows: Given a box beam such as shown in figure 1 subjected to the action of torques concentrated at certain bulkheads, find the stresses caused by the torques.

The cross sections are assumed to be rectangular and doubly symmetrical. The shapes of the cross sections are assumed to be maintained by ribs or bulkheads rigid in their own planes. The derivation of the formulas is facilitated somewhat if the actual cross sections of the box are transformed into simplified cross sections of the type

shown in figure 2. The walls of the simplified sections are assumed to carry only shear stresses τ , the corner flanges only normal stresses σ . The transformation implies no assumptions other than standard ones on stress distribution and is discussed in the appendix.

The box beam under torque loads is a statically indeterminate structure. A statically determinate structure is obtained by cutting the redundant members at the stations where the torques are applied and where the dimensions of the cross section change, thus dividing the box into a number of bays. The four flanges are chosen as redundant members in accordance with Ebner's method (reference 1). For reasons of static equilibrium, the redundant forces form a doubly antisymmetrical group of four forces X at each station (fig. 3). Under the action of the torques T and the forces X , each individual bay deforms as indicated by the dashed lines in figure 3; the magnitudes of the forces X are calculated by the principle of consistent deformations of adjacent bays.

Two sets of formulas are given for the stresses in and the deformations of an individual bay. The first set is based on assumption A that no intermediate ribs exist within the bay; the second set is based on assumption B that closely spaced intermediate ribs exist within the bay. The formulas obtained under assumption B are more general and reduce to the corresponding formulas obtained under assumption A in the limit as a characteristic parameter K_a appearing in the formulas approaches zero. The approximate method of analysis described at the end of the paper is based on assumption B.

Sign conventions and notations.— External torques T are positive when acting clockwise viewed from the tip. Forces X are positive when acting in the directions shown in figure 3. Shear stresses τ are positive when acting in the direction of shear stresses caused by positive torques. Normal stresses σ are positive when caused by positive X -forces. The warping deformations w are positive in the directions indicated by the dashed lines in figure 3, that is, in the directions of positive X -forces acting on the outboard ends of the bays.

A coordinate x is needed only for formulas dealing with an individual bay. The origin is taken at the outboard end of the bay under consideration (fig. 3).

Bays and stations are numbered as shown in figure 4. Superscripts identify the force causing the stress or the deformation whenever definite identification is desirable. Subscripts 1 and o denote the inboard and outboard ends, respectively.

Stresses in an Individual Bay

Each individual bay is acted upon by three independent sets of loads: a torque T on each end, a group of forces X_1 at the inboard end, and a group of forces X_o at the outboard end. Formulas will be given for the stresses caused by each load individually; the final stresses are obtained in each case by superposing the stresses caused by the three sets of loads.

Stresses caused by torque.— The shear stresses set up by the torque acting on a given bay are given by the familiar formula for shells in torsion and are, with the notation of figure 2,

$$\left. \begin{aligned} \tau_b &= \frac{T}{2bct_b} \\ \tau_c &= \frac{T}{2bct_c} \end{aligned} \right\} \quad (1)$$

The formulas are valid for bays without intermediate ribs (assumption A) as well as for bays with intermediate ribs (assumption B). No stresses σ are set up in the flanges when a bay is loaded by pure torques.

Stresses caused by X-forces in a bay without intermediate ribs.— A group of X-forces acting on one end of a bay sets up normal stresses σ in the flanges and shear stresses τ in the walls. The sign convention adopted makes it necessary to distinguish a group X_1 applied at the inboard end and a group X_o applied at the outboard end of the bay.

In a bay without intermediate bulkheads, the shear stresses caused by an X-group cannot vary spanwise because spanwise variations of the shear stress in a bay can be effected only by bulkheads transferring shear stresses from one pair of walls to the other pair. The

shear stresses being constant; the flange forces vary linearly along the bay (fig. 5), and the flange stresses caused by an inboard group and by an outboard group of X-forces are given, respectively, by the formulas

$$\sigma = \frac{xX_1}{aA} \quad (2)$$

$$\sigma = \frac{(a-x)X_0}{aA} \quad (3)$$

The shear stresses are obtained by applying two equations of equilibrium. The condition $\Sigma T = 0$ applicable to any cross section of the bay gives

$$\tau_b t_b bc + \tau_c t_c bc = 0 \quad (4)$$

The condition $\Sigma X = 0$ applied to a flange acted upon by an X-force at the inboard end gives

$$X_1 + \tau_b t_b a - \tau_c t_c a = 0 \quad (5)$$

The combination of equations (4) and (5) yields the result

$$\left. \begin{aligned} \tau_b &= -X_1/2at_b \\ \tau_c &= X_1/2at_c \end{aligned} \right\} \quad (6)$$

When the forces X are applied at the outboard end of the bay, the stresses are

$$\left. \begin{aligned} \tau_b &= X_0/2at_b \\ \tau_c &= -X_0/2at_c \end{aligned} \right\} \quad (7)$$

Stresses caused by X-forces in a bay with intermediate ribs.— In a bay with intermediate ribs, the shear stress varies spanwise (fig. 6). The equation of equilibrium $\Sigma X = 0$ for the flange must therefore be written in the form

$$A \frac{d\sigma}{dx} + \tau_b t_b - \tau_c t_c = 0 \quad (8)$$

which yields on differentiation

$$A \frac{d^2 \sigma}{dx^2} + t_b \frac{d\tau_b}{dx} - t_c \frac{d\tau_c}{dx} = 0 \quad (9)$$

The shear strains of the walls are obtained by adding the strains caused by twisting and the strains caused by warping. If θ denotes the angle of twist,

$$\left. \begin{aligned} \gamma_b &= + \frac{c}{2} \frac{d\theta}{dx} + \frac{2w}{b} \\ \gamma_c &= + \frac{b}{2} \frac{d\theta}{dx} - \frac{2w}{c} \end{aligned} \right\} \quad (10)$$

Elimination of $\frac{d\theta}{dx}$ from equation (10), differentiation, and multiplication by the shear modulus G gives

$$b \frac{d\tau_b}{dx} - c \frac{d\tau_c}{dx} = -4G \frac{dw}{dx} \quad (11)$$

Now $\frac{dw}{dx} = -\frac{\sigma}{E}$ by fundamental definition, where E is Young's modulus, and $\tau_c t_c = -\tau_b t_b$ by equation (4). Substitution of these expressions in equation (11) and then in equation (9) yields the differential equation

$$\frac{d^2 \sigma}{dx^2} - \frac{8G\sigma}{AE(b/t_b + c/t_c)} = 0 \quad (12)$$

With the boundary conditions $\sigma = 0$ at $x = 0$ and $\sigma = X/A_1$ at $x = a$, valid for an X-group acting at the inboard end of the bay, the solution of the differential equation is

$$\sigma = \frac{X_1 \sinh Kx}{A \sinh Ka} \quad (13)$$

and by analogy for an X-group at the outboard end

$$\sigma = \frac{X_0 \sinh K(a-x)}{A \sinh Ka} \quad (14)$$

where the parameter K is defined by the equation

$$K^2 = \frac{8G}{AE(b/t_b + c/t_c)} \quad (15)$$

The shear stresses caused by an X-group acting on the bay may now be found by substituting the value of σ given by formula (13) or (14) in formulas (8) and (4); they are

$$\left. \begin{aligned} \tau_b &= - \frac{X_1 K \cosh Kx}{2t_b \sinh Ka} \\ \tau_c &= \frac{X_1 K \cosh Kx}{2t_c \sinh Ka} \end{aligned} \right\} \quad (16)$$

$$\left. \begin{aligned} \tau_b &= \frac{X_0 K \cosh K(a-x)}{2t_b \sinh Ka} \\ \tau_c &= - \frac{X_0 K \cosh K(a-x)}{2t_c \sinh Ka} \end{aligned} \right\} \quad (17)$$

Deformations of an Individual Bay

Principle of calculation.— Under the action of the torques and of the groups of X-forces, the end cross sections of an individual bay warp out of their original planes (fig. 8). The magnitude w of the warping will be calculated by the method variously called method of internal work, method of dummy unit loading, method of virtual work, etc. This method involves three operations: First, the stresses σ and τ caused by the applied loads are calculated. Second, a force or a system of forces U (unit force) is applied in the direction of the deformation sought, and the stresses σ^U and τ^U caused by the force U are calculated. Third, the deformation is calculated by the principle that the external work done by the unit force must equal the internal energy stored by virtue of the existence of the unit force; this principle is stated in the equation

$$\sum U w = \iiint \frac{\sigma \sigma^U}{E} dV + \iiint \frac{\tau \tau^U}{G} dV \quad (18)$$

where V is the volume of the stressed material; the summation sign designates that U may be a group of forces.

The warping of a cross section is doubly antisymmetrical (fig. 3). The dummy unit loadings employed in this particular problem consist therefore of doubly antisymmetrical groups of four forces U similar to the X -groups chosen as the redundancies of the statically indeterminate system. The stresses caused by the dummy unit loads U can therefore be calculated by the formulas for stresses caused by X -forces; care must be taken, however, to use the signs in agreement with the sign conventions.

Warping caused by torque.— The stresses caused by a torque T acting on a bay are by equation (1)

$$\left. \begin{aligned} \sigma &= 0 \\ \tau_b &= \frac{T}{2bct_b} \\ \tau_c &= \frac{T}{2bct_c} \end{aligned} \right\} \quad (19)$$

In order to compute the warping at the outboard end of the bay, a dummy unit load consisting of four forces U is introduced at the outboard end. Under the assumption that no intermediate ribs exist in the bay, the stresses σ^U and τ^U caused by these U -forces can be computed by substituting U for X_0 in formulas (3) and (7). The results are

$$\left. \begin{aligned} \sigma^U &= \frac{U(a-x)}{Aa} \\ \tau_b^U &= U/2at_b \\ \tau_c^U &= -U/2at_c \end{aligned} \right\} \quad (20)$$

The expressions for σ , τ , σ^U , and τ^U given by formulas (19) and (20) are now substituted in equation (18) and give

$$\begin{aligned} 4Uw_0 &= 2 \int_{x=0}^{x=a} \frac{T}{G 2bct_b} \frac{U}{2at_b} bt_b dx \\ &\quad + 2 \int_{x=0}^{x=a} \frac{1}{G 2bct_c} \left(-\frac{U}{2at_c} \right) ct_c dx \end{aligned}$$

which yields on integration

$$w_o^T (= w_1^T = w^T) = \frac{T}{8Gb_o} \left(\frac{b}{t_b} - \frac{c}{t_c} \right) \quad (21)$$

This derivation of the formula was given for the specific case of warping at the outboard end of a bay without intermediate ribs. The formula is not restricted to this case, however; it applies to bays without or with intermediate ribs, and it gives the warping at the inboard end as well as at the outboard end. This fact can be verified easily by substituting the proper stresses τ^U in equation (18) and integrating; it can also be deduced directly from the fact that the shear stresses caused by a torque are not affected by intermediate ribs.

Warping caused by X-forces.— The warping at the outboard end of a bay caused by an X-group acting at the outboard end may be written in the form

$$w_o = pX_o \quad (22)$$

where the coefficient p is obtained by applying equation (18) in the form

$$4Uw_o = 4 \int_{x=0}^{x=a} \frac{\sigma_o^U}{E} A \, dx + 2 \int_{x=0}^{x=a} \frac{\tau_b \tau_b^U}{G} b t_b \, dx + 2 \int_{x=0}^{x=a} \frac{\tau_c \tau_c^U}{G} c t_c \, dx$$

By substituting in this equation the proper formulas for the stresses and integrating, the coefficient p is found to be, for a bay without intermediate ribs,

$$p = \frac{a}{3AE} + \frac{1}{8Ga} \left(\frac{b}{t_b} + \frac{c}{t_c} \right) \quad (23)$$

and, for a bay with intermediate ribs,

$$\begin{aligned}
p &= \frac{1}{2KA\bar{E}} \left(\coth Ka - \frac{Ka}{\sinh^2 Ka} \right) \\
&\quad + \frac{K}{16G} \left(\coth Ka + \frac{Ka}{\sinh^2 Ka} \right) \left(\frac{b}{t_b} + \frac{c}{t_c} \right) \\
&= \frac{\coth Ka}{KA\bar{E}}
\end{aligned} \tag{24}$$

The warping at the inboard end caused by an X-group at the inboard end is

$$w_1 = -pX_1 \tag{25}$$

where p is given by expression (23) or (24), as the case may be.

The warping at the inboard end of a bay caused by an X-group acting at the outboard end of the bay may be written in the form

$$w_1 = qX_0 \tag{26}$$

By substituting the proper stress formulas in equation (18) and integrating, the coefficient q is found to be, for a bay without intermediate ribs,

$$q = -\frac{a}{6A\bar{E}} + \frac{1}{8Ga} \left(\frac{b}{t_b} + \frac{c}{t_c} \right) \tag{27}$$

and, for a bay with intermediate ribs,

$$\begin{aligned}
q &= -\frac{1}{2KA\bar{E}} \left(-\frac{1}{\sinh Ka} + \frac{Ka \cosh Ka}{\sinh^3 Ka} \right) \\
&\quad + \frac{K}{16G} \left(\frac{1}{\sinh Ka} + \frac{Ka \cosh Ka}{\sinh^3 Ka} \right) \left(\frac{b}{t_b} + \frac{c}{t_c} \right) \\
&= \frac{1}{KA\bar{E} \sinh Ka}
\end{aligned} \tag{28}$$

The warping at the outboard end of a bay caused by an X-group acting at the inboard end is given by the expression

$$w_0 = -qX_1 \tag{29}$$

The general formula (24) for a bay with many ribs reduces to the special formula (23) for a bay without intermediate ribs in the limit when the parameter Ka approaches zero. Similarly, formula (28) reduces to formula (27) when Ka approaches zero. When the parameter Ka becomes very large, formulas (24) and (28) simplify to

$$p = \frac{1}{2KA\bar{E}} + \frac{K}{16G} \left(\frac{b}{t_b} + \frac{c}{t_c} \right) = \frac{1}{KA\bar{E}} \quad (30)$$

$$q = 0 \quad (31)$$

Formulas (30) and (31) are sufficiently accurate for most practical purposes when $Ka > 5$ and may be used as approximations when $Ka > 3$.

In order to simplify the numerical evaluation of formulas (24) and (28), they may be written in the form

$$p = \frac{a}{3A\bar{E}} C' + \frac{1}{8G_n} \left(\frac{b}{t_b} + \frac{c}{t_c} \right) C'' \quad (32)$$

$$q = - \frac{a}{6A\bar{E}} D' + \frac{1}{8Ga} \left(\frac{b}{t_b} + \frac{c}{t_c} \right) D'' \quad (33)$$

The coefficients C' , C'' , D' , and D'' are given in figure 7.

Recurrence Formula for the Calculation of X-Forces

Derivation of recurrence formula.— According to the principle of consistent deformations, the warping at the inboard end of one bay must be equal to the warping of the adjacent outboard end of the next bay. The warping at the inboard end of bay n , that is, at station n , equals

$$w_{n1} = w_n^T - p_n X_n + q_n X_{n-1} \quad (34)$$

In this expression, the subscripts n and $(n-1)$ of the forces X designate the stations at which the forces act, whereas the subscript n of w , p , and q designates the bay under consideration. The warping at the outboard end of bay $n+1$ equals

$$w_{(n+1)0} = w_{n+1}^T + p_{n+1} X_n - q_{n+1} X_{n+1} \quad (35)$$

Equating formulas (34) and (35) and rearranging the terms yields the recurrence formula

$$q_n X_{n-1} - (p_n + p_{n+1}) X_n + q_{n+1} X_{n+1} = -w_n^T + w_{n+1}^T \quad (36)$$

By giving n successive values from $n = 1$ to $n = r$, there is obtained a set of equations, each of which contains three of the redundant force groups X . There is one equation for each station except station 0 at the tip.

Boundary conditions.— The tip of a box beam is normally free from axial loads or restraints; the forces X at the tip are therefore zero, and the first equation of the system is

$$-(p_1 + p_2) X_1 + q_2 X_2 = -w_1^T + w_2^T \quad (37)$$

When a beam is attached to a rigid foundation (fig. 1), the foundation may be considered a bay $(r + 1)$ having infinite shear stiffness and infinite axial stiffness, therefore

$$w_{r+1}^T = p_{r+1} = q_{r+1} = 0$$

and the last equation of the system becomes

$$q_r X_{r-1} - p_r X_r = -w_r^T \quad (38)$$

The condition of a rigid foundation may exist in a practical structure by virtue of symmetry (fig. 8).

The torque reaction is frequently located at a distance d from the plane of symmetry (fig. 9), and the forces X_r are transmitted from the root of one wing to the root of the opposite wing by carry-through members having axial stiffnesses AE . The last equation of the system for this case can be written by inspection as

$$q_r X_{r-1} - p_r' X_r = -w_r^T \quad (39)$$

where

$$p_r' = p_r + \frac{d}{AE} \quad (40)$$

ANALYSIS OF A BOX WITH FULL-WIDTH CUT-OUTS

General considerations.— In actual box beams, full-width cut-outs are often necessary (fig. 10). If the region of the cut-out is considered a bay and if the standard procedure is used for rendering the structure statically determinate by cutting each flange at each station, it will be found that the remaining structure is incapable of carrying load. The cut-out region can transmit torque only by two-spar or beam action of the two active walls. (The wall opposite the cut-out is inactive because it can carry no shear loads for reasons of static equilibrium.) Beam action requires active participation of the flanges; it is therefore not permissible to cut them at both ends of the cut-out region, and the procedure of rendering the structure determinate must be modified. The modification consists in combining the cut-out region with the adjoining complete box section either on the inboard side or on the outboard side into a combination bay (fig. 11) and cutting the flanges at the ends of the combination bay. As indicated by figure 11, a combination bay will be termed "type I" when the two-spar part is outboard and type II when the two-spar part is inboard.

Stresses and deformations of a combination bay, type I.

In the two-spar part of a combination bay of type I (fig. 11), the stresses are found by statics and are given by the equations

$$\sigma = \frac{xT}{Abc} + \frac{X_0}{A} \quad (C \text{ to } D) \quad (41)$$

$$\tau_c = \frac{T}{bct_0} \quad (C \text{ to } D) \quad (42)$$

The formulas for the stresses in the box part of a combination bay and for the warping of the entire bay depend on the construction of the box part and will be given as before under assumption A (only end bulkheads exist) and assumption B (many intermediate ribs exist). Under assumption A, the formulas for the stresses are

$$\sigma = \left(\frac{Td}{Abc} + \frac{X_0}{A} \right) \left(1 - \frac{x}{a} \right) + \frac{xX_1}{aA} \quad (D \text{ to } E) \quad (43)$$

$$\tau_b = \frac{T}{2bct_b} \left(1 + \frac{d}{a}\right) + \frac{X_0}{2at_b} - \frac{X_1}{2at_b} \quad (D \text{ to } E) \quad (44)$$

$$\tau_c = \frac{T}{2bct_c} \left(1 - \frac{d}{a}\right) - \frac{X_0}{2at_c} + \frac{X_1}{2at_c} \quad (D \text{ to } E) \quad (45)$$

Under assumption B, the formulas for the stresses are

$$\sigma = \left(\frac{Td}{Abc} + \frac{X_0}{A}\right) \frac{\sinh K(a-x)}{\sinh Ka} + \frac{X_1}{A} \frac{\sinh Kx}{\sinh Ka} \quad (D \text{ to } E) \quad (46)$$

$$\begin{aligned} \tau_b = \frac{T}{2bct_b} \left[1 + Kd \frac{\cosh K(a-x)}{\sinh Ka} \right] + \frac{X_0 K}{2t_b} \frac{\cosh K(a-x)}{\sinh Ka} \\ + \frac{X_1 K}{2t_b} \frac{\cosh Kx}{\sinh Ka} \quad (D \text{ to } E) \end{aligned} \quad (47)$$

$$\begin{aligned} \tau_c = \frac{T}{2bct_c} \left[1 - Kd \frac{\cosh K(a-x)}{\sinh Ka} \right] - \frac{X_0 K}{2t_c} \frac{\cosh K(a-x)}{\sinh Ka} \\ + \frac{X_1 K}{2t_c} \frac{\cosh Kx}{\sinh Ka} \quad (D \text{ to } E) \end{aligned} \quad (48)$$

The formulas for the warping of a combination bay may be written in a general form valid under assumption A as well as under assumption B by using the coefficients p and q previously introduced. Warping caused by torque is given by the formulas

$$w_0^T = w^T + p \frac{Td}{bc} + \frac{Td^2}{2bcAE} \quad (49)$$

$$w_1^T = w^T + q \frac{Td}{bc} \quad (50)$$

The quantity w^T is calculated by formula (21). The coefficients p and q are calculated for the box part of the combination bay by the proper formula for assumption A or B, as the case may be. The terms with p and q in formulas (49) and (50) arise from the fact that the

torque applied at the two-spar end creates an X-group having a magnitude Td/bc at the junction between the two-spar part and the box part. The last term in formula (49) represents the deformation of the flanges in the two-spar part; the values of A and B should, therefore, be understood to be the average values in the two-spar region.

The formulas for warping caused by X-groups are

$$w_0 = p'X_0 - qX_1 \quad (51)$$

$$w_1 = qX_0 - pX_1 \quad (52)$$

where p' is given by formula (40).

Stresses and deformations of a combination bay, type II.— For a combination bay of type II (fig. 11), the formulas for shear stresses are the same as for a bay of type I. The formulas for flange stresses and for warping are replaced by the following formulas:

$$\sigma = -\frac{Td}{Abc} \left(1 - \frac{x}{d}\right) + \frac{X_1}{A} \quad (D \text{ to } E) \quad (53)$$

$$\sigma = \left(-\frac{Td}{Abc} + \frac{X_1}{A}\right) \frac{x}{a} + \frac{X_0}{A} \left(1 - \frac{x}{a}\right) \quad (C \text{ to } D) \quad (54)$$

$$\sigma = \left(-\frac{Td}{Abc} + \frac{X_1}{A}\right) \frac{\sinh Kx}{\sinh Ka} + \frac{X_0}{A} \frac{\sinh K(a-x)}{\sinh Ka} \quad (C \text{ to } D) \quad (55)$$

$$w_0^T = w^T + q \frac{Td}{bc} \quad (56)$$

$$w_1^T = w^T + p \frac{Td}{bc} + \frac{Td^2}{2bcAE} \quad (57)$$

$$w_0 = pX_0 - qX_1 \quad (58)$$

$$w_1 = qX_0 - pX_1 \quad (59)$$

Modifications of the recurrence formula.— The particular nature of a combination bay makes it necessary to

modify slightly the two equations in which the deformation at one end of the combination bay is equated to the deformation of the adjoining bay. On the assumption that the combination bay is the m th bay of the beam, the modified equations are, for a bay of type I,

$$q_{m-1}X_{m-2} - (p_{m-1} + p_m') X_{m-1} + q_m X_m = -w_{m-1}^T + w_{m0}^T \quad (60)$$

$$q_m X_{m-1} - (p_m + p_{m+1}) X_m + q_{m+1} X_{m+1} = -w_{m1}^T + w_{m+1}^T \quad (61)$$

and, for a bay of type II,

$$q_{m-1}X_{m-2} - (p_{m-1} + p_m) X_{m-1} + q_m X_m = -w_{m-1}^T + w_{m0}^T \quad (62)$$

$$q_m X_{m-1} - (p_m' + p_{m+1}) X_m + q_{m+1} X_{m+1} = -w_{m1}^T + w_{m+1}^T \quad (63)$$

The differences from the standard form consist in the appearance of the term p' instead of p in two places and in the appearance of two distinct terms for the torque warping at the outboard end and at the inboard end of the combination bay.

APPROXIMATE METHOD OF ANALYSIS

Calculations on typical wing structures have shown that the bending stresses due to torsion are seldom more than about one-tenth the direct bending stresses. Consequently, an accuracy of 10 percent in the calculation of bending stresses due to torsion will give an accuracy of about 1 percent on the total stresses, which is ample for stress analysis. In most practical cases, then, the simplified method of analysis described here is sufficiently accurate; cases in which a more accurate analysis by the general method is advisable can be recognized by inspection. Reference may be made to the section entitled "General Considerations" for sign conventions and other preliminaries.

Approximate analysis of a box without cut-outs.— The actual cross sections of the box (fig. 1) are transformed into idealized sections (fig. 2) by the method discussed

in the appendix. The major discontinuities of the cross-sectional dimensions and of the torque loading are located by inspection (wing root, locations of wing engines, location of landing gear). For any given discontinuity, the quantities K , p , and w^T are computed by the formulas

$$K^s = \frac{8G}{AE(b/t_b + c/t_c)} \quad (15)$$

$$p = \frac{G}{2KA^s} + \frac{K}{16} \left(\frac{b}{t_b} + \frac{c}{t_c} \right) = \frac{G}{KA^s} \quad (30')$$

$$w^T = \frac{T}{8bc} \left(\frac{b}{t_b} - \frac{c}{t_c} \right) \quad (21')$$

where T is the applied torque and the superscript T indicates that the warping w is caused by the torque; the meaning of the other nonstandard symbols is explained by figure 2. Two values of each of these quantities are calculated: one, denoted by the subscript o , for the region just outboard of the discontinuity, and one, denoted by the subscript i , for the region just inboard of the discontinuity. For a root section rigidly fixed, or for the central plane of a symmetrical box loaded symmetrically (fig. 8), the quantities p_i and w_i^T become zero.

The flange forces at the discontinuity are calculated by the formula

$$X = \frac{w_o^T - w_i^T}{p_o + p_i} \quad (64)$$

from which the flange stresses follow as $\sigma = X/A$. The shear stresses caused by the discontinuity are calculated by the formulas

$$\left. \begin{aligned} \tau_b &= -\frac{XK}{2t_b} \\ \tau_c &= \frac{XK}{2t_c} \end{aligned} \right\} \quad (65)$$

To those shear stresses caused by the discontinuity must be added the shear stresses caused by the direct action of

the torque, which are given by the basic formula for shells in torsion

$$\left. \begin{aligned} \tau_b &= \frac{T}{2bct_b} \\ \tau_c &= \frac{T}{2bct_c} \end{aligned} \right\} \quad (1)$$

The values of the stresses σ and τ at a distance x from the discontinuity are obtained by multiplying the stresses given by formulas (64) and (65) by the factor e^{-Kx} . This factor may be used in the following manner to indicate whether the approximate method is sufficiently accurate in a given case.

The approximate method is based on the assumption that the stresses caused by a given discontinuity are negligible at the location of the next discontinuity. If an accuracy of 10 percent is considered sufficient as suggested, the approximate theory is sufficiently accurate when $e^{-Ka} < 0.1$, where a is the distance between two successive discontinuities along the span. In practice, the specified relative accuracy need be maintained only for the maximum stresses. The criterion $e^{-Ka} < 0.1$ need, therefore, be met only for the region between the discontinuity causing the largest bending stresses due to torsion (usually the root of the wing) and the nearest discontinuity.

The flange forces X at the root of a wing with a carry-through bay (fig. 9) are obtained from the equation

$$\left(p + \frac{dG}{AE}\right) X = w^T \quad (66)$$

where d is the length of a carry-through member as defined in figure 9 and A is its cross-sectional area.

Approximate analysis of a box with cut-out.— When the discontinuity being investigated is the junction between a box region and a region with a full-width cut-out (two-spar region), formula (64) must be replaced by more complicated formulas containing the properties of the two-spar region as well as the properties of both adjoining

box regions. Two formulas are necessary, one giving the forces X_0 at the outboard end of the two-spar region, the other one giving the forces X_1 at the inboard end of the two-spar region. The formulas are

$$- \left(p_0 + p_1 + \frac{dG}{AE} \right) X_0 = -w_0^T + w_1^T + p_1 \frac{Td}{bc} + \frac{Td^2G}{2bcAE} \quad (67)$$

$$- \left(p_0 + p_1 + \frac{dG}{AE} \right) X_1 = -w_0^T + w_1^T - p_0 \frac{Td}{bc} - \frac{Td^2G}{2bcAE} \quad (68)$$

where d is the length of the cut-out and A the area of a corner flange in this region.

It may be noted that formulas (30') and (21') differ by the factor G from the corresponding formulas (30) and (21). The modified form given here is obtained by eliminating the factor $1/G$ and replacing the factor $1/E$ by G/E . This procedure is permissible if it is applied to each of the quantities p , p' , q , and w , because the factor $1/G$ can be canceled on both sides of the formula for the flange forces. The modified form is more suitable for numerical work than the original form.

NUMERICAL EXAMPLES

The numerical examples will be based on a box beam with the following properties:

b , inches	60
c , inches	10
t_b , inch	0.040
t_c , inch	0.080
A , square inches	3.00
Total length, inches	300
G/E	0.385

A torque of 120,000 pound-inches is applied in all cases. For examples of beams with variable cross section, the values of t_b , t_c , and A in the outboard half of the beam will be assumed to be half as large as the basic values just given. In order to simplify the calculation of the quantities w , p , p' , and q , the factor $1/G$ will al-

ways be omitted, and the factor $1/E$ will be replaced by G/E as discussed in the preceding section.

Example 1.— The box is built in at the root and has only one bulkhead at the tip. No intermediate ribs exist. The torque is applied at the tip. Find the stresses in the box.

The entire box constitutes a single bay, because no bulkheads or ribs exist except at the tip. There is only one unknown X_1 which acts at the root. For a single unknown, the system of equations reduces to the single equation

$$-X_1 p_1 = -w_1 T$$

The value of $w_1 T$ is, by formula (21),

$$w_1 T = \frac{120000}{8 \times 60 \times 10} \left(\frac{60}{0.040} - \frac{10}{0.080} \right) = 34,375$$

The value of p_1 is, by formula (23),

$$p_1 = \frac{300 \times 0.385}{3 \times 3.00} + \frac{1}{8 \times 300} \left(\frac{60}{0.040} + \frac{10}{0.080} \right) = 13.51$$

The solution is

$$X_1 = \frac{34375}{13.51} = 2540 \text{ pounds}$$

$$\sigma = \frac{2540}{3.00} = 847 \text{ pounds per square inch}$$

From the maximum value at the root, the flange stresses decrease linearly to zero at the tip.

The basic shear stresses caused by the torque are, by formula (1),

$$\tau_b = \frac{120000}{2 \times 60 \times 10 \times 0.040} = 2500 \text{ pounds per square inch}$$

$$\tau_c = \frac{120000}{2 \times 60 \times 10 \times 0.080} = 1250 \text{ pounds per square inch}$$

The shear stresses caused by the X-force are, by formula (6),

$$\tau_b = - \frac{2540}{2 \times 300 \times 0.040} = - 106 \text{ pounds per square inch}$$

$$\tau_c = \frac{2540}{2 \times 300 \times 0.080} = 53 \text{ pounds per square inch}$$

The final shear stresses are therefore

$$\tau_b = 2500 - 106 = 2394 \text{ pounds per square inch}$$

$$\tau_c = 1250 + 53 = 1303 \text{ pounds per square inch}$$

Example 2.— The box is divided into four bays by bulkheads, but no intermediate ribs exist. The torque is applied at the tip. Find the stresses in the box.

By formula (21),

$$w_1^T = w_2^T = w_3^T = w_4^T = 34,375 \text{ (as in example 1)}$$

By formula (23),

$$p_1 = p_2 = p_3 = p_4 = \frac{75 \times 0.385}{3 \times 3.00} + \frac{1}{8 \times 75} \left(\frac{60}{0.040} + \frac{10}{0.080} \right) = 5.91$$

By formula (27),

$$q_1 = q_2 = q_3 = q_4 = - \frac{75 \times 0.385}{6 \times 3.00} + \frac{1}{8 \times 75} \left(\frac{60}{0.040} + \frac{10}{0.080} \right) = 1.102$$

The system of equations for X is therefore:

$$- (5.91 + 5.91)X_1 + 1.102X_2 = -34,375 + 34,375$$

$$1.102X_1 - (5.91 + 5.91)X_2 + 1.102X_3 = -34,375 + 34,375$$

$$1.102X_2 - (5.91 + 5.91)X_3 + 1.102X_4 = -34,375 + 34,375$$

$$1.102X_3 - 5.91X_4 = -34,375$$

The solution of this system is

$$X_1 = 5 \text{ pounds}$$

$$X_2 = 52 \text{ pounds}$$

$$X_3 = 556 \text{ pounds}$$

$$X_4 = 5920 \text{ pounds}$$

The shear stresses in the root bay caused by the X -forces are, by formulas (6) and (7),

$$\tau_b = - \frac{5920}{2 \times 75 \times 0.040} + \frac{556}{2 \times 75 \times 0.040} = - 894 \text{ pounds per square inch}$$

$$\tau_c = \frac{5920}{2 \times 75 \times 0.080} - \frac{556}{2 \times 75 \times 0.080} = 447 \text{ pounds per square inch}$$

The final shear stresses in the root bay are therefore

$$\tau_b = 2500 - 894 = 1606 \text{ pounds per square inch}$$

$$\tau_c = 1250 + 447 = 1697 \text{ pounds per square inch}$$

Figure 12 shows graphically the spanwise variation of the flange forces for this beam and for beams divided into $n = 1, 2$, and 3 bays, as well as for a beam with many bulkheads, which will be analyzed in the next example. It may be noted that the flange force at the root obtained in this example for a bulkhead spacing of 75 inches differs only by about 12 percent from the corresponding value for the box with infinitely many bulkheads.

Example 3.— The box has a tip bulkhead and many intermediate ribs. The torque is applied at the tip. Find the stresses in the box.

The entire box is considered a single bay. The value of w^T is the same as in the preceding examples. By formula (15),

$$K^2 = \frac{8 \times 0.385}{3 \times 1625} = 0.000632$$

therefore

$$K = 0.0251$$

$$K_a = 7.53$$

By formula (30),

$$p = \frac{0.385}{2 \times 0.0251 \times 3.00} + \frac{0.0251 \times 1625}{16} = 5.11$$

and, by formula (22),

$$X_1 = \frac{w^T}{p} = \frac{34375}{5.11} = 6740 \text{ pounds}$$

The spanwise variation of the X-force is calculated by formula (13)

$$X = X_1 \frac{\sinh Kx}{\sinh Ka}$$

The shear stresses are obtained by combining the shear stresses caused by torque given by formula (1) and the shear stresses caused by X_1 given by formulas (16). Figure 13 shows the stresses in graphical form.

Example 4.— The box has end bulkheads and many intermediate ribs. The torque is applied at midspan. Find the stresses in the box.

By formula (21),

$$w_1^T = 0$$

$$w_2^T = 34,375 \quad (\text{See example 1.})$$

From example 3,

$$K = 0.0251$$

$$Ka = 0.0251 \times 150 = 3.76$$

By formula (24),

$$p_1 = p_2 = \frac{0.385}{2 \times 0.0251 \times 3.00} \left[\frac{1}{0.999} - \frac{3.76}{(21.5)^2} \right] + \frac{0.0251 \times 1625}{16} \left[\frac{1}{0.999} + \frac{3.76}{(21.5)^2} \right] = 5.11$$

By formula (28),

$$q_1 = q_2 = - \frac{0.385}{2 \times 0.0251 \times 3.00} \left[- \frac{1}{21.5} + \frac{3.76 \times 21.5}{(21.5)^2} \right] \\ + \frac{0.0251 \times 1625}{16} \left[\frac{1}{21.5} + \frac{3.76 \times 21.5}{(21.5)^2} \right] = 0.238$$

The system of equations obtained by applying the recurrence formula is therefore

$$- (5.11 + 5.11)X_1 + 0.238X_2 = -0 + 34,375$$

$$0.238X_1 - 5.11X_2 = -34,375 + 0$$

The solution is $X_1 = -3510$ pounds

$$X_2 = 6560 \text{ pounds}$$

Figure 14 shows the final stresses in graphical form.

A comparison of figures 13 and 14 indicates that the stresses at the root change very little when the spanwise location of the torque changes from the tip to midspan. The local stress peaks at the midspan torque are much lower than the peaks caused at the root by the same torque.

Examples 5, 6, and 7.— For the following three examples, only the final results are shown in graphical form. It is assumed in all three examples that there are many intermediate ribs.

Figure 15 (example 5) shows the stresses in a box of constant cross section with many ribs, subjected to the action of five equal torques evenly spaced along the span. The sum of the five torques is equal to the torque of 120,000 pound-inches used in the previous examples. The stresses at the root are nearly equal to those calculated in examples 3 and 4, showing again that the root stresses depend chiefly on the total torque at the root and very little on the distribution of this torque.

Figure 16 (example 6) shows the stresses in a box under tip torque when the thickness t_b and t_c and the flange

area A in the outboard half are reduced to one-half their basic values. The discontinuity in the dimensions causes local stress peaks.

Figure 17 (example 7) shows the stresses in the box of the preceding example when the torque is applied at midspan.

Example 8.— The box of example 3 is attached to a carry-through bay (fig. 9); the length of this bay is $d = 30$ inches, the cross-sectional area of each member in it is $A = 3.00$ square inches. Find the X -force at the root.

From example 3,

$$p = 5.11$$

$$w^T = 34,375$$

$$q = 0$$

By formula (40),

$$p' = 5.11 + \frac{30 \times 0.385}{3.00} = 8.96$$

By formula (39),

$$-8.96X_1 = -34,375$$

$$X_1 = 3840 \text{ pounds}$$

Comparison with example 3 shows that the presence of the carry-through bay reduces the maximum flange force by about 40 percent.

Example 9.— The box of example 3 has the top cover and the bottom cover removed over the region from $x = 150$ to $x = 180$ inches from the tip. Find the X -forces along the span.

The two-spar region will be combined with the region between it and the root to form a combination bay of type I. By formulas (24) and (28),

$$p_1 = 5.11$$

$$q_1 = 0.224$$

$$p_2 = 5.06$$

$$q_2 = 0.502$$

By formula (40),

$$p_2' = 8.91$$

By formulas (49) and (50),

$$w_{20} = 34,375 + 30,350 + 11,540 = 76,265$$

$$w_{21} = 34,375 + 3,010 = 37,385$$

By formulas (60) and (61),

$$-(5.11 + 8.91)X_1 + 0.502X_2 = -34,375 + 76,265$$

$$0.505X_1 - 5.10X_2 = -37,385$$

The solution of these equations is

$$X_1 = -2735 \text{ pounds}$$

$$X_2 = 7120 \text{ pounds}$$

The X-force at the junction between the two-spar part and the box part of the combination bay is obtained from formula (41) after multiplying through by A:

$$X = \frac{30 \times 120000}{60 \times 10} - 2735 = 3265 \text{ pounds}$$

The spanwise distribution of the X-force is shown in figure 18.

Example 10.— Solution of example 9 by the approximate method.

By formula (15), as in example 3,

$$K = 0.0251$$

By formula (30'),

$$p = \frac{0.385}{2 \times 0.0251 \times 3.00} + \frac{0.0251}{16} \left(\frac{60}{0.040} + \frac{10}{0.080} \right) = 5.11$$

By formula (21'),

$$w^T = \frac{120000}{8 \times 60 \times 10} \left(\frac{60}{0.040} - \frac{10}{0.080} \right) = 34,375$$

Furthermore,

$$\frac{dG}{dT} = \frac{30 \times 0.385}{3.00} = 3.85$$

$$\frac{Td}{bc} = \frac{120000 \times 30}{60 \times 10} = 6000$$

$$p \frac{Td}{bc} = 30,660$$

$$\frac{Td^2G}{2bcAE} = 11,550$$

At the root, by formula (64), with $p_1 = 0$

$$-5.11X = -34,375$$

$$X = 6725 \text{ pounds}$$

At the outboard end of the two-spar region, by formula (67),

$$- (5.11 + 5.11 + 3.85)X_0 = -34,375 + 34,375 + 30,660 + 11,550$$

$$X_0 = -3000 \text{ pounds}$$

Similarly at the inboard end, by formula (68),

$$X_1 = 3000 \text{ pounds}$$

In this particular case, X_1 is equal to X_0 because the box inboard of the two-spar region has the same dimensions and the same torque loading as the box outboard of the two-spar region.

At a station 50 inches from the root, the value of X is reduced by the factor e^{-Kx} :

$$X_{50} = 6725e^{-1.255} = 1915 \text{ pounds}$$

At the same station, there is a contribution from X_1 . The distance from X_1 equals 70 inches; therefore,

$$X_{50} = 3000e^{-1.757} = 518 \text{ pounds}$$

The total force at this station is, therefore,

$$X_{50} = 1915 + 518 = 2433 \text{ pounds}$$

The values of X obtained by the approximate method are shown as circles in figure 18.

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APPENDIX

TRANSFORMATION OF ACTUAL BEAM SECTIONS INTO IDEALIZED SECTIONS

If the web of a plate girder is assumed to furnish no contribution to the bending strength of the girder, the section modulus of the girder is given by the expression hA_F , where A_F is the cross-sectional area of a flange. The section modulus of the web acting alone is given by the expression $\frac{1}{6}h^3t$. The section modulus of the entire girder is

$$Z = h\left(A_F + \frac{1}{6}ht\right)$$

The actual girder may therefore be replaced for the purpose of computing extreme fiber stresses by a fictitious girder having a web carrying only shear stresses and a flange having an area equal to $\left(A_F + \frac{1}{6}ht\right)$. This substitution is especially useful when four girders are combined to form a box, because the condition of continuity of stresses along the edges is then automatically fulfilled. The fictitious flange area becomes, for this case, $A_F + \frac{1}{6}bt_b + \frac{1}{6}ct_c$.

When longitudinal stringers are rivoted to the web, the same substitution may be used if it is understood to mean the cross-sectional area of all material effective in bonding, exclusive of the concentrated flanges themselves. Care must be taken, however, to use reduced areas where the stringers are interrupted by cut-outs.

The thickness of the fictitious web capable of carrying only shear stresses may be made equal to the actual thickness of the web. This method is approximate, but it is sufficiently accurate in most cases. When the web forms diagonal-tension fields, an appropriate correction must be made to the shear modulus.

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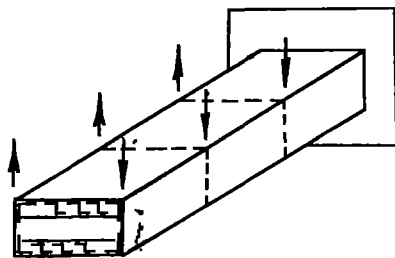


Figure 1.- Box beam under torque loads.

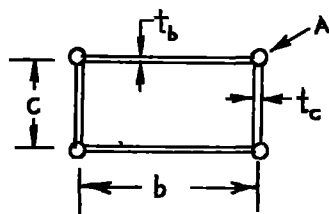


Figure 2.-Idealized cross section of box.

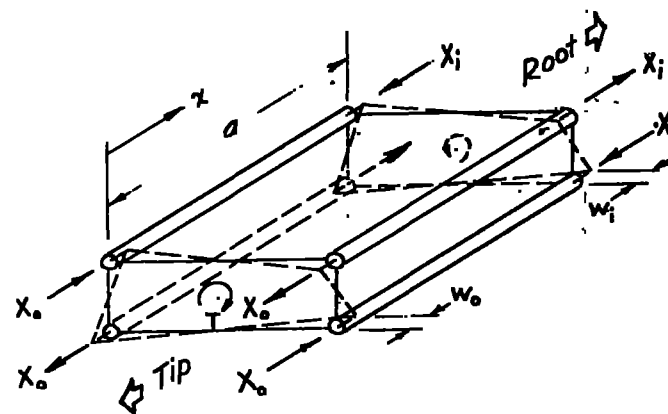


Figure 3.- Free-body sketch of individual bay.

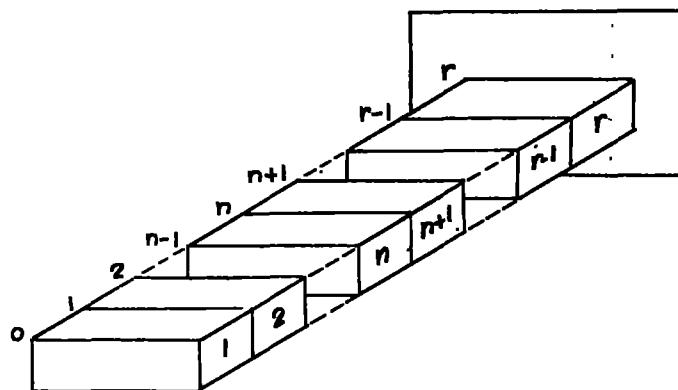


Figure 4.- Convention for numbering bays and stations.

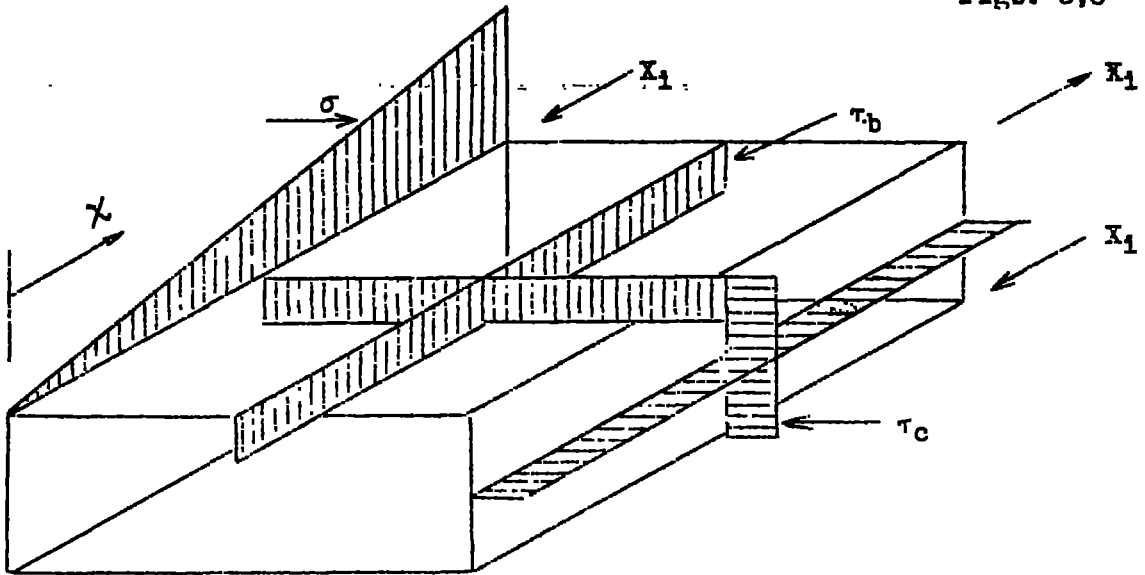


Figure 5.- Stresses caused by X_1 -group in a bay without intermediate ribs.

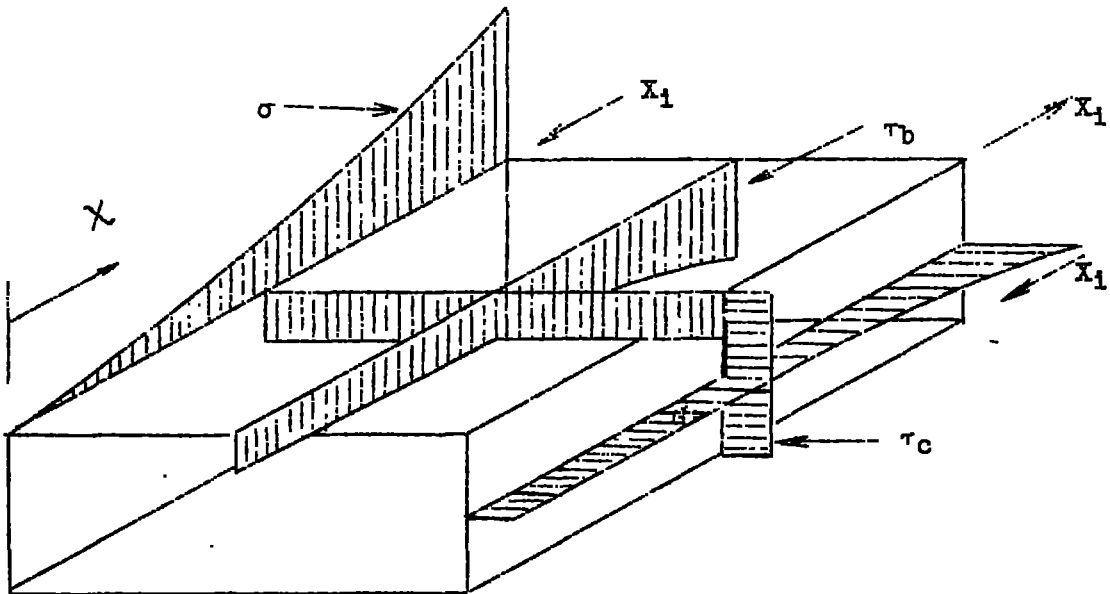


Figure 6.- Stresses caused by X_1 -group in a bay with intermediate ribs.

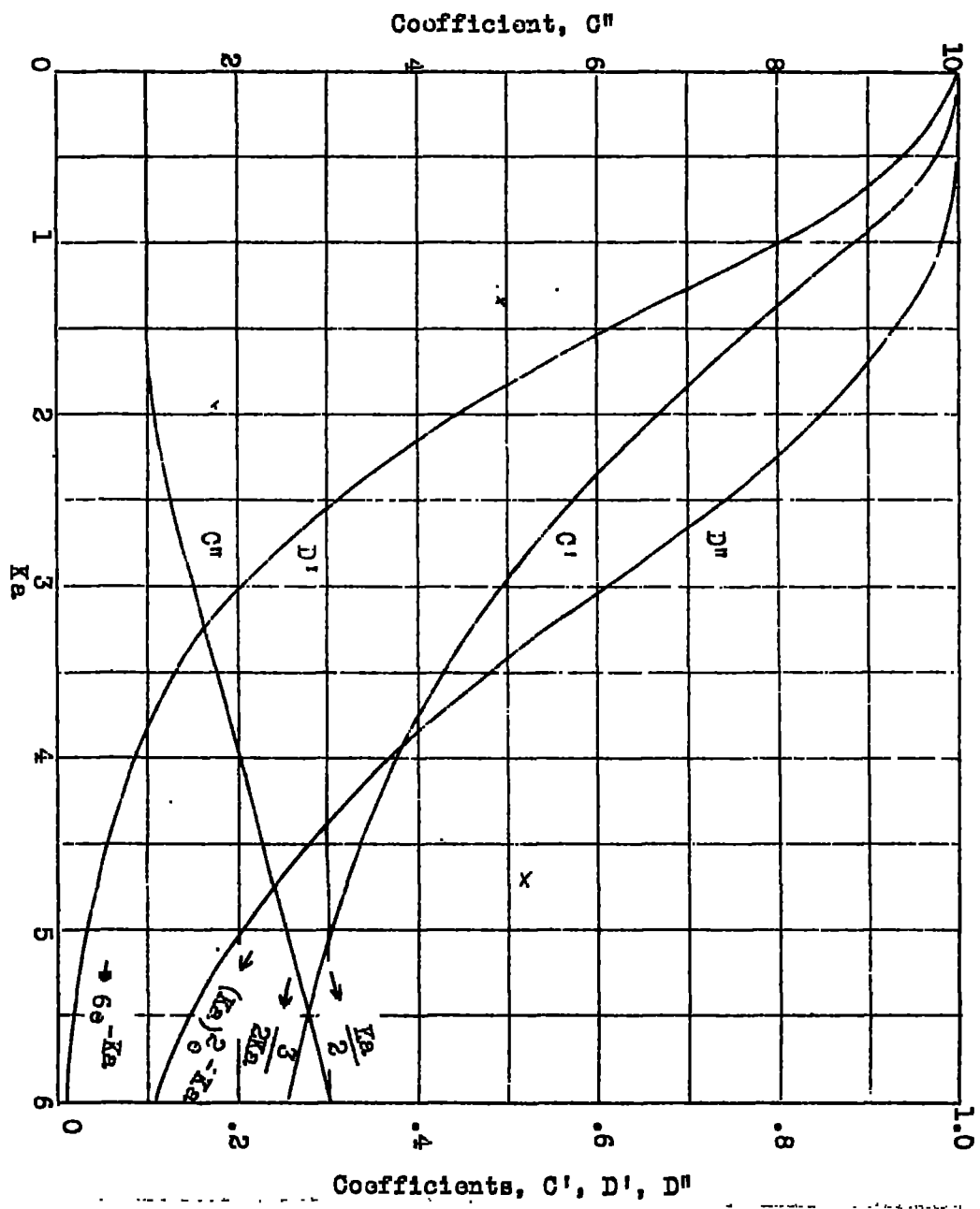


Figure 7.- Coefficients C' , C'' , D' , and D'' .

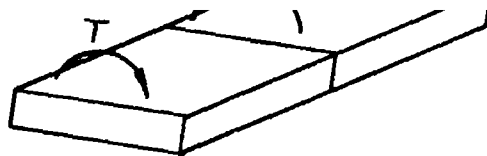


Figure 8.- Symmetrical box.

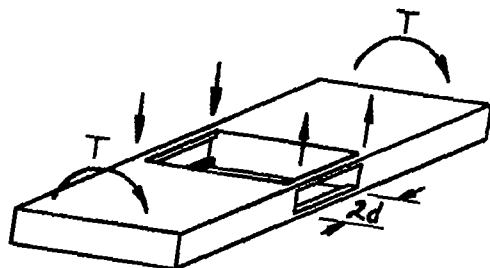


Figure 9.- Symmetrical box beam with carry-through bay at center.

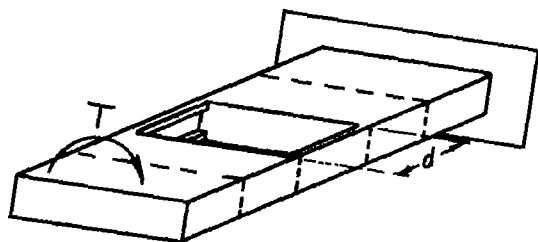
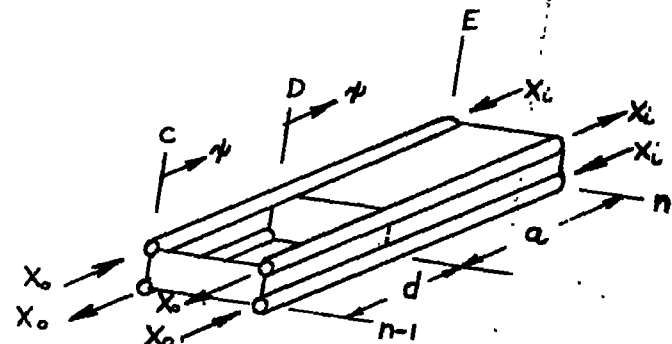
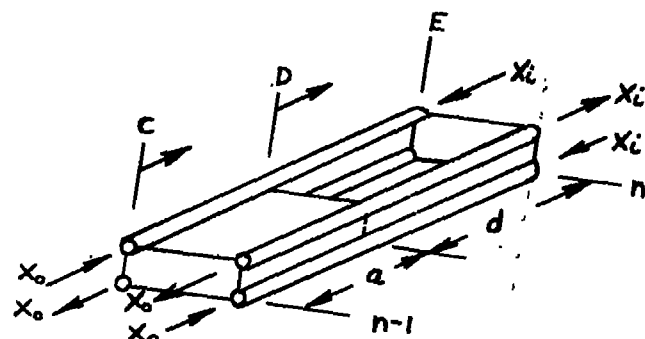


Figure 10.- Box beam with full-width cut-out.



(a) Type I.



(b) Type II.

Figure 11.- Combination bays.

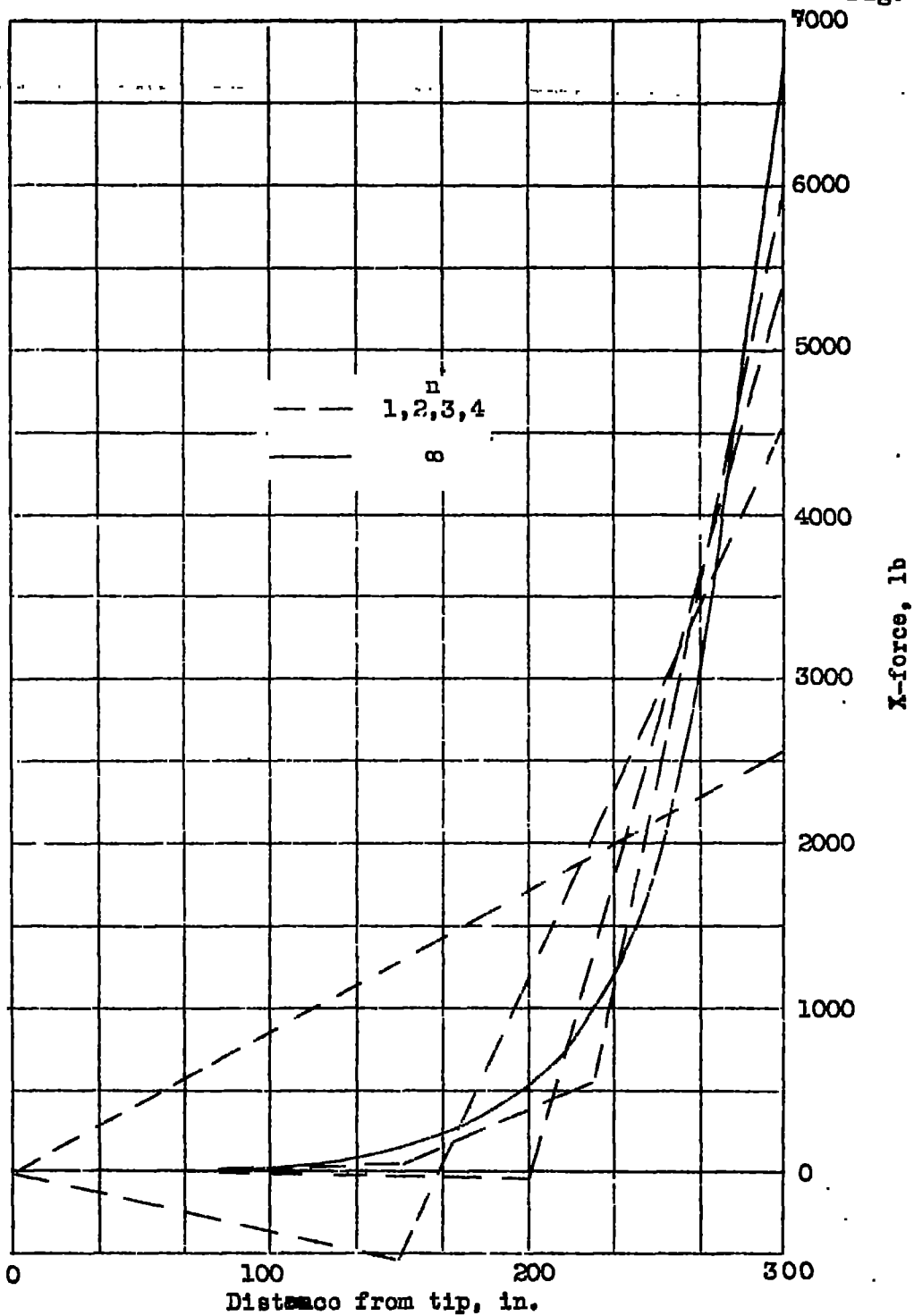


Figure 12.- Spanwise distribution of X-forces for $n = 1, 2, 3, 4, \infty$ bays.

L-552

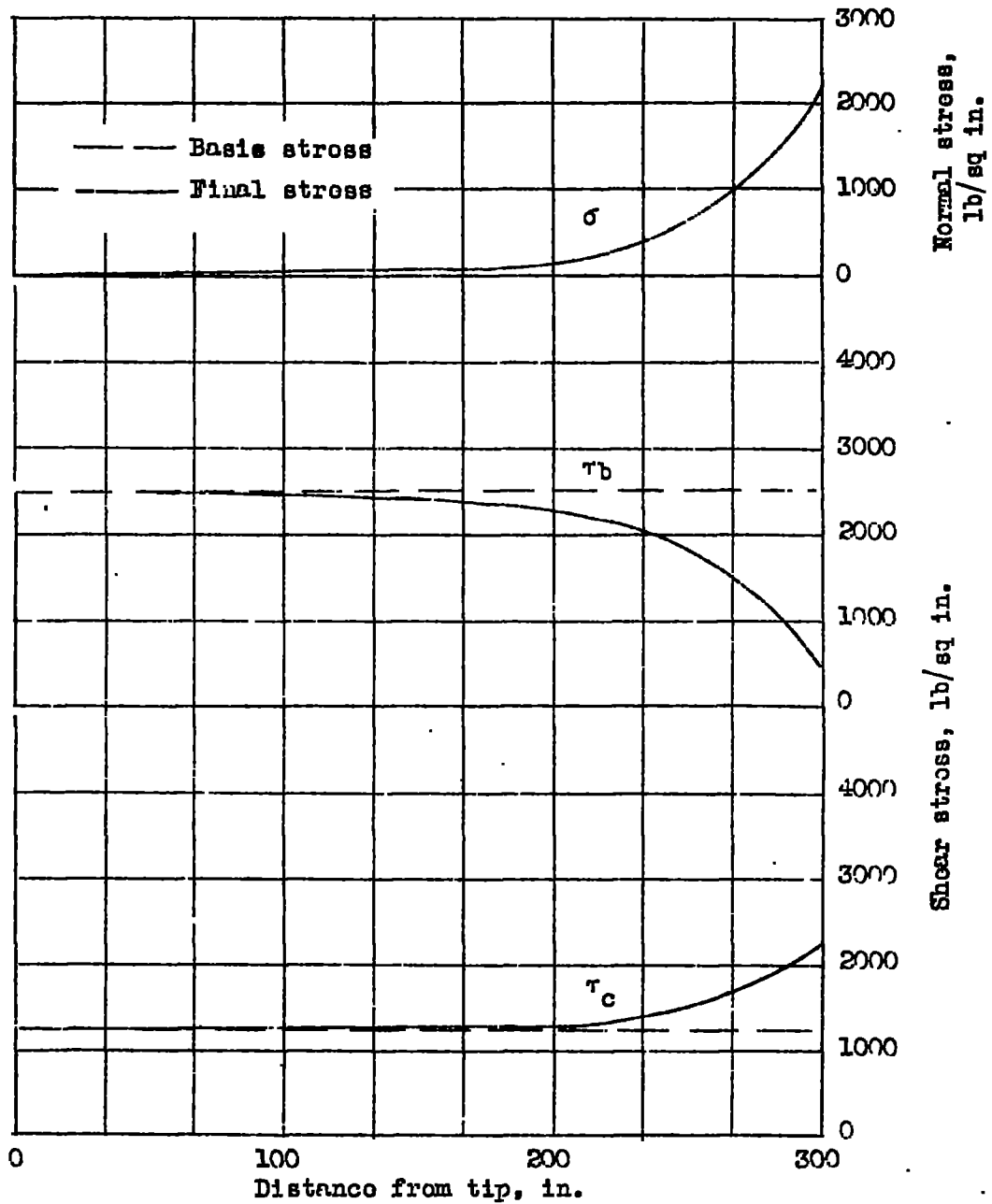


Figure 13.- Stresses in box of constant cross section with torque applied at tip. Example 3.

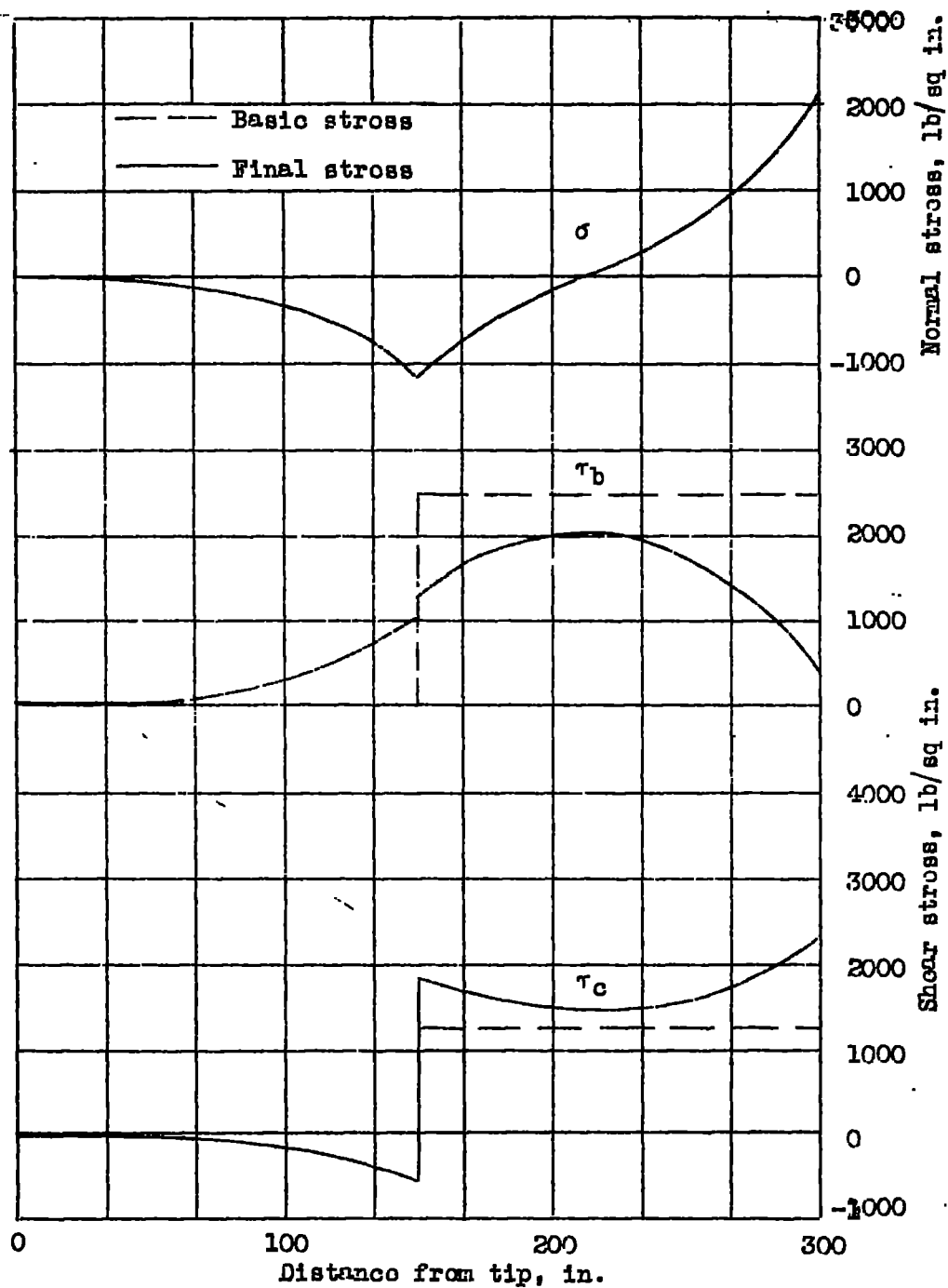


Figure 14.- Stresses in box of constant cross section with torque applied at midspan. Example 4.

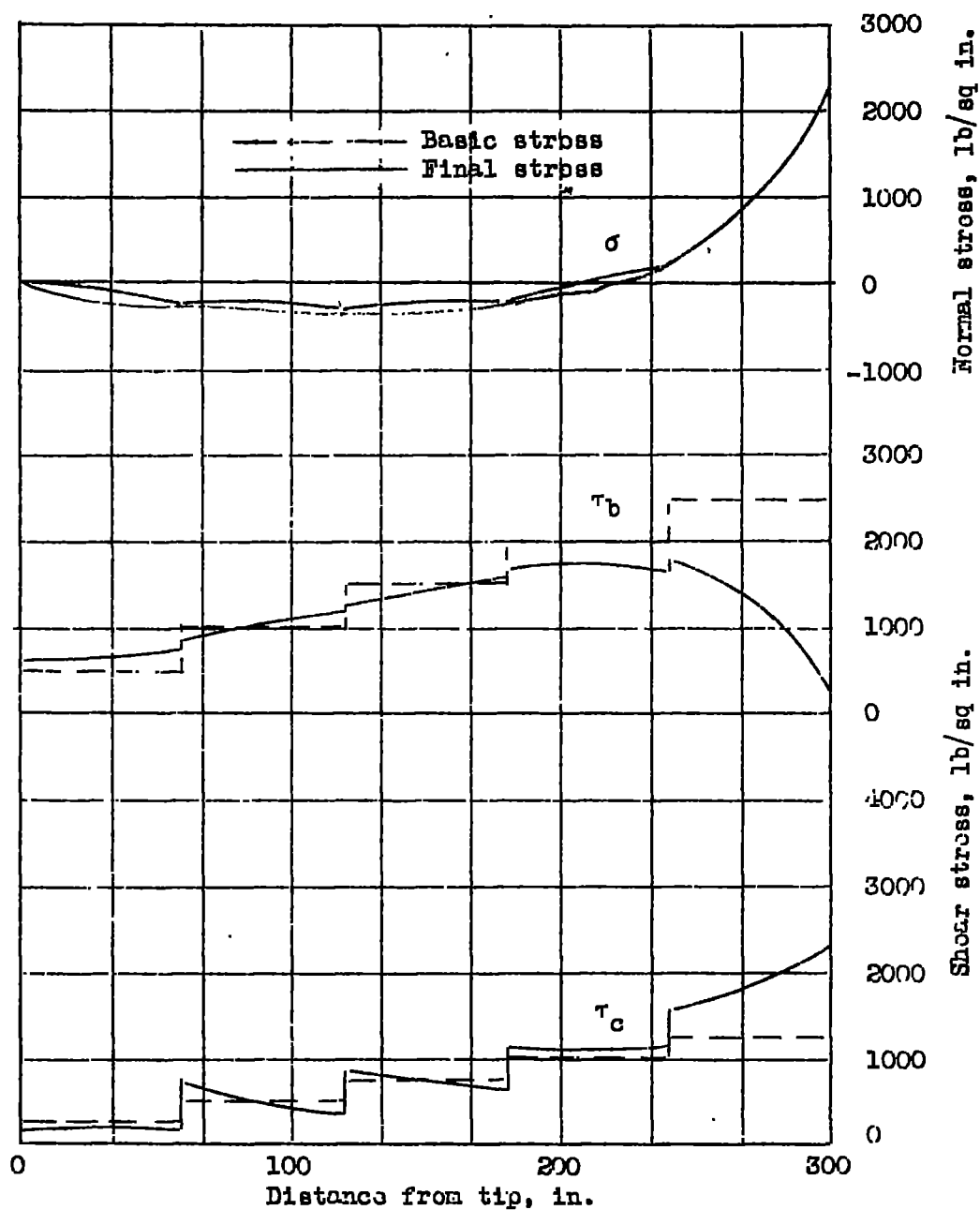


Figure 15.- Stresses in box of constant cross section with torque distributed over 5 stations. Example 5.

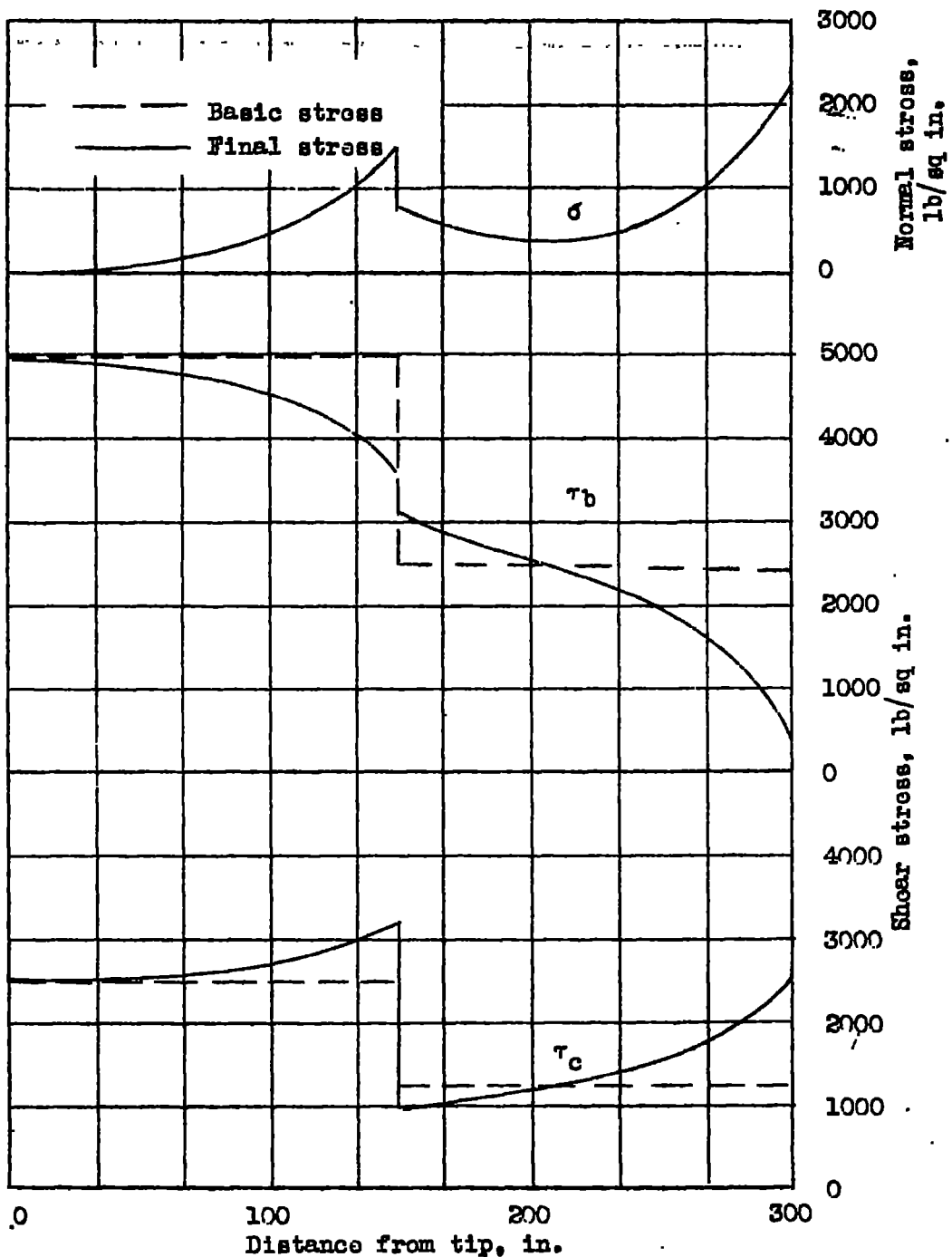


Figure 16.-- Stresses in box of variable cross section with torque applied at tip. Example 6.

L-352

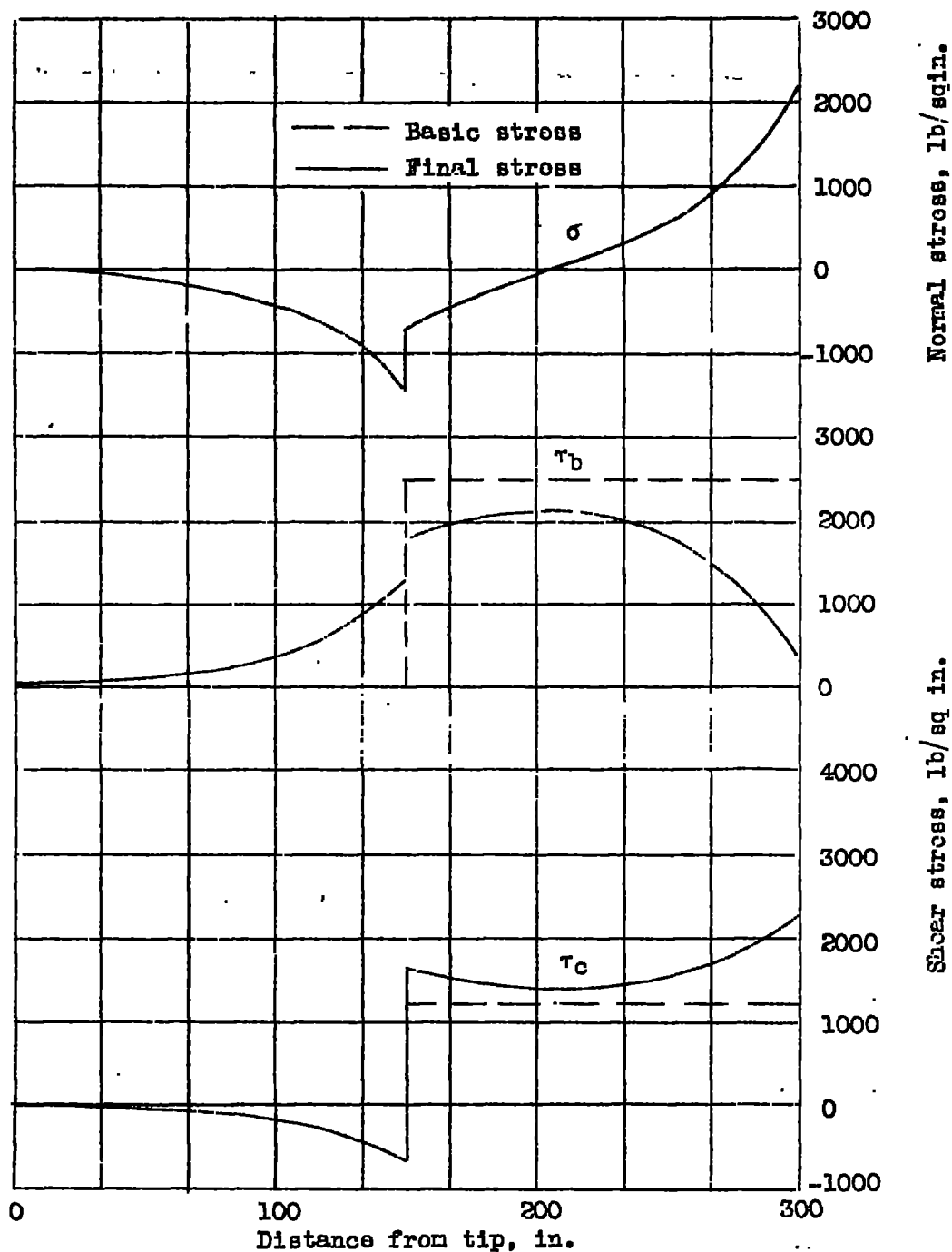


Figure 17.- Stresses in box of variable cross section with torque applied at midspan. Example 7.

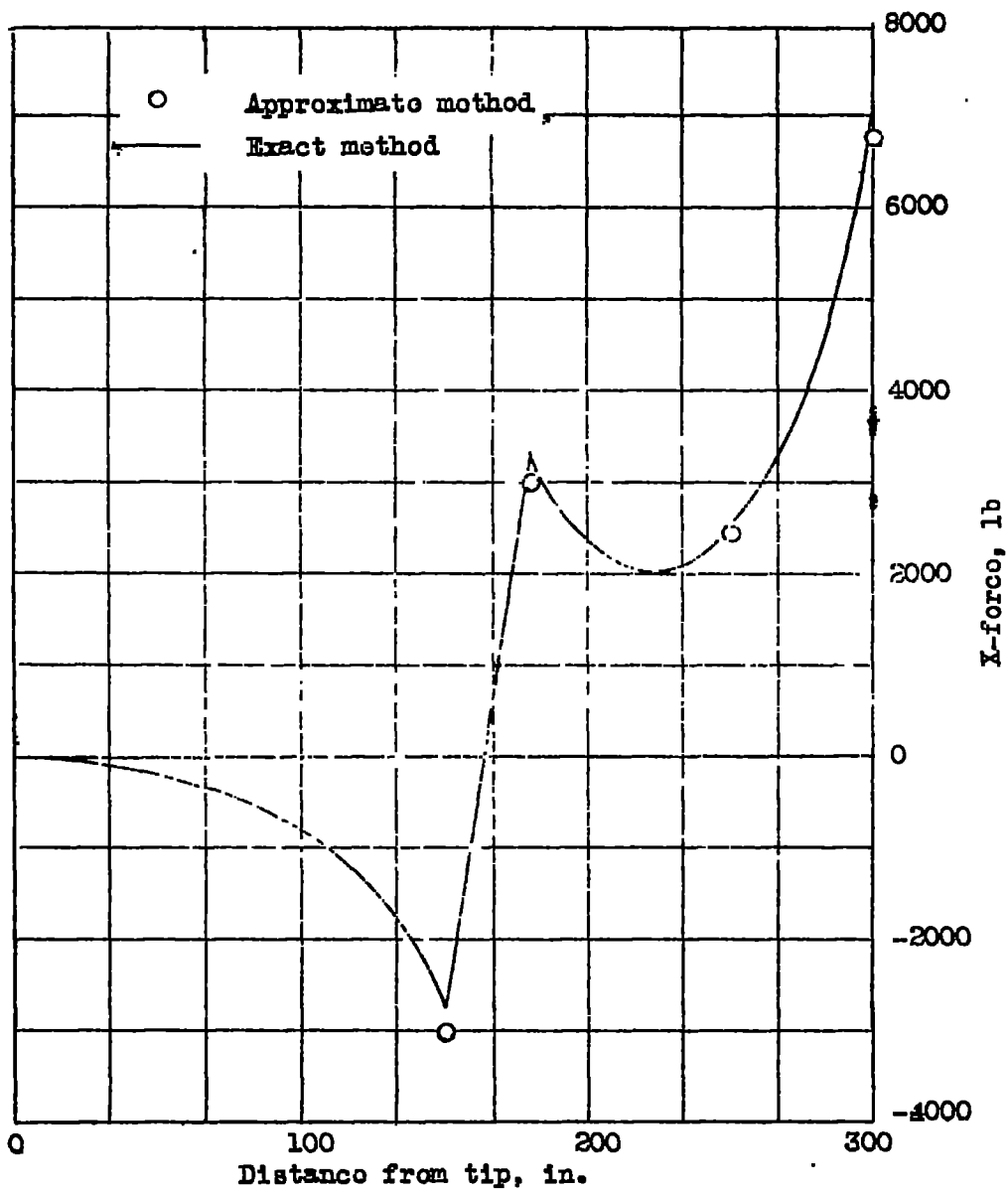


Figure 18.-- X-forces in box with cut-out; torque applied at tip. Examples 9 and 10.